Trace anomaly, thermal power corrections
and dimension two condensates in the deconfined phase

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The trace anomaly of gluodynamics on the lattice shows clear fingerprints of a dimension two condensate above the phase transition. The condensate manifests itself through even powers of the inverse temperature while the total perturbative contribution corresponds to a mild temperature dependence and turns out to be compatible with zero within errors. We try several resummation methods based on a renormalization group improvement. The trace anomaly data are analyzed and compared with other determinations of the dimension two condensate based on the Polyakov loop and the heavy $q\bar{q}$ free energy, yielding roughly similar numerical values. The role of glueballs near the transition is also discussed.

I. INTRODUCTION

The physics of the Quark-Gluon Plasma has turned out to be more difficult than initially expected (for a recent review see e.g. [1] and references therein). While it has been noted [2] that the failure of the perturbative description could have been anticipated on general grounds, it has not been obvious what is the form and furthermore the dynamical origin of such a strongly non-perturbative behaviour. On the other hand it is by now clear from lattice calculations at finite temperature (for a review on recent developments see e.g. [3] and references therein) that while in QCD there is a cross-over, in pure Yang-Mills theory or gluodynamics for $N_c \geq 3$ there is a first order phase transition. Actually, in the simpler case of gluodynamics the deconfinement phase transition is monitored by the expectation value of the Polyakov loop which acts as an order parameter of the associated breaking of the center $\mathbb{Z}(N_c)$ symmetry.

The finding of power corrections above the phase transition [4] and the possible explanation in terms of a dimension-2 condensate of the dimensionally reduced theory of such a special object suggests pursuing the same idea in other thermodynamic quantities. Power corrections are ubiquitous at high energies and zero temperature through condensates being the remnant of non-perturbative effects. From this viewpoint there is no reason why they should not be present at high temperatures as genuinely non-perturbative effects. In the present paper we detect and analyze power corrections on the trace anomaly density at finite temperature as obtained in lattice calculations, and which can only be effectively explained by a dimension two condensate. Unlike the Polyakov loop, the trace anomaly corresponds to a physical and measurable quantity not only on the lattice but also experimentally in the QCD case. In this way we extend our study of the non-perturbative effects above the phase transition started for the Polyakov loop and the $q\bar{q}$ free energy [4, 5, 6]. A brief account of our results have been presented in Refs. [7, 8].

The paper is organized as follows. In Section II we provide mounting evidence on the dominance of power corrections for the trace anomaly using available lattice data for gluodynamics [9]. The thermodynamics of the phase transition is analyzed in Section III following an interesting insight by Pisarski [10]. The physics below the phase transition in the confined region should be describable in terms of colour singlet glueball states, an issue that we address within a large N_c perspective where the interactions among glueballs are much suppressed. In passing we note a Hagedorn looking glueball exponentially growing spectrum with practically no impact on the confined region. We analyze in Section IV our results from the point of view of analytical schemes based on perturbation theory. Actually, a common deficiency of the poorly converging perturbation theory is the occurrence of unpleasant infrared divergences and manifest spurious scale dependence. Perturbative resummations such as Hard Thermal Loop remove the infrared singularities due to Debye screening but provide too weak a signal as compared to the lattice data. Our analysis suggests that those methods might in fact describe the deconfined phase *after* the power corrections have been subtracted off. To reinforce this conclusion a renormalization group improvement of the perturbative free energy is undertaken in this Section. We find that besides implement-

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ing renormalization scale independence, this resummation mimics the Hard Thermal Loop calculations within estimated uncertainties, and as a consequence similar conclusion follows: power corrections dominate the deconfined phase while perturbatively based physics plays a minor role wherever it may be considered applicable. The dimension two condensate has naturally explained the presence of power corrections for the Polyakov loop and singlet $q\bar{q}$ free energy [4, 6]. The idea is tried out in Section V for the trace anomaly where we indeed show that there is a satisfactory numerical agreement for the dimension two condensate with that of the Polyakov loop and the $q\bar{q}$ free energy. Finally in Section VI we give our main conclusions. In Appendix A we rederive the trace anomaly at finite temperature for reference. In Appendix B the possible Hagedorn pattern of the glueball spectrum is analyzed. In Appendix C we study an improvement of the perturbative expansion of the free energy at finite temperature based on renormalization group invariance.

II. THERMAL POWER CORRECTIONS IN THE TRACE ANOMALY

For zero or infinite quark masses (gluodynamics) the classical scale invariance of QCD is broken by quantum corrections due to the necessary regularization yielding a trace anomaly [11]. In the context of QCD sum rules phenomenology the issue has been discussed in Ref. [12] (for a review see e.g. [13]). At finite temperature the trace of the energy momentum tensor is related not only to the trace anomaly but also to the difference $\epsilon - 3P$ with ϵ the energy density and P the pressure [14] (see also Ref. [15, 16]) providing a measure of the interaction as well as the anomalous breaking of scale invariance. An early lattice calculation of both energy and pressure was first undertaken one decade ago [9, 17] (see also more recent calculations [18, 19, 20, 21]) exhibiting the expected deconfinement phase transition although far still from a free gluonic plasma. Finite temperature Ward identities and low energy theorems based on the trace anomaly have been deduced in Refs. [22, 23] in the continuum and extended to Euclidean lattices [24]. Renormalization issues and the identity of thermodynamic and hydrostatic pressure was discussed in Ref. [25] within the real time formulation. Phenomenological implications in the context of resonance gas model are analyzed in [26]. A good knowledge of the trace anomaly is crucial to understand the deconfinement process, where the non perturbative (NP) nature of low energy QCD seems to play a prominent role [2].

The derivation of the well known relation

$$\epsilon - 3P = \frac{\beta(g)}{2g}\langle (G_{\mu\nu}^a)^2 \rangle, \qquad (2.1)$$

between the trace anomaly and the beta function of gluodynamics is reviewed in Appendix A.

The trace anomaly density has been computed at finite temperature on the lattice for pure gluodynamics [9]. The standard plot of

$$\Delta \equiv \frac{\epsilon - 3P}{T^4} = T\frac{\partial}{\partial T}\left(\frac{P}{T^4}\right) \qquad (2.2)$$

as a function of T is shown in Fig. 1. Below the critical temperature the trace anomaly Δ is very small though not exactly zero. Δ increases suddenly near and above T_c by latent heat of deconfinement, and raises a maximum at $T \approx 1.1\,T_c$. Then it has a gradual decrease reaching zero in the high temperature limit. The high value of Δ for $T_c \leq T \leq 3\,T_c$ corresponds to a strongly interacting Quark-Gluon Plasma picture.

In previous works [4, 6] we have detected the presence of inverse power corrections on other thermal observables. Guided by our previous experience we represent in Fig. 2 the dimensionless interaction measure $(\epsilon - 3P)/T^4$ as a function of $1/T^2$ (in units of T_c). This plot exposes an unmistakable straight line behaviour corresponding to a power correction of the form

$$\frac{\epsilon - 3P}{T^4} = a_\Delta + b_\Delta\left(\frac{T_c}{T}\right)^2 \qquad (2.3)$$

in the deconfined phase region slightly above the critical temperature. A direct fit of the lattice data ($N_\sigma^3 \times N_\tau = 16^3 \times 4$) for $1.13 < T/T_c < 5.0$ yields

$$a_\Delta = -0.041(17), \quad b_\Delta = 3.99(5), \quad \chi^2/\text{dof} = 5.0 \quad (2.4)$$

with correlation parameter $r(a, b) = -0.78$. Likewise, the fit for ($N_\sigma^3 \times N_\tau = 32^3 \times 8$) and $1.13 < T/T_c < 4.5$ yields

$$a_\Delta = -0.02(4), \quad b_\Delta = 3.46(13), \quad \chi^2/\text{dof} = 0.35 \quad (2.5)$$

with $r(a, b) = -0.73$. The fit is more accurate in the second set of lattice data. This set corresponds to a more precise determination of the trace anomaly on the lattice. In order to estimate the result in the continuum limit, we can assume that the difference between the two lattice results is entirely due to finite cutoff effects. Assuming further that the corresponding leading effect goes as $1/N_\tau^2$, yields the estimate $a_\Delta = -0.02(9)$ and $b_\Delta = 3.28(27)$ in the continuum limit.

A fit of the data completely excludes the existence of the odd power corrections T_c/T and $(T_c/T)^3$ in Δ. We have attempted to determine the coefficient of a possible quartic correction, appending formula (2.3) with a term $c_\Delta\,(T_c/T)^4$. A fit of lattice data for $N_\tau = 8$ results in $a_\Delta = -0.05(7)$, $b_\Delta = 3.7(7)$ and $c_\Delta = -0.3(9)$ with $\chi^2/\text{dof} = 0.31$ and correlations $r(a, b) = -0.85$, $r(b, c) = -0.97$ and $r(a, c) = 0.75$. So, more accurate data are desirable in order to identify contributions from condensates of dimension 4.

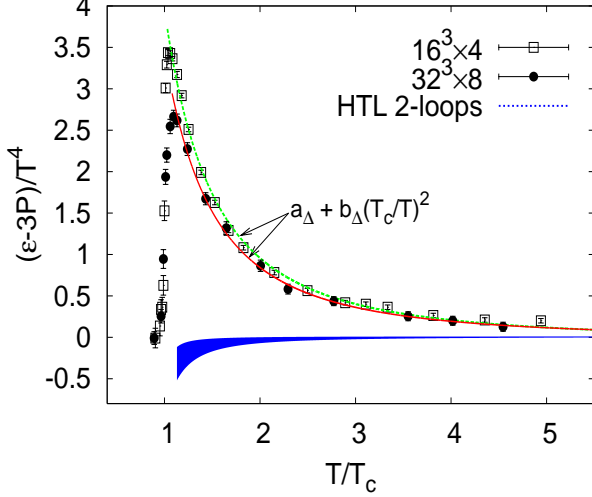In any case it is clear the existence of power corrections which are beyond the scope of the radiative corrections

FIG. 1: The trace anomaly density $(\epsilon - 3P)/T^4$ as a function of T (in units of T_c). Lattice data are from [9] for $N_\sigma^3 \times N_\tau = 16^3 \times 4$ and $32^3 \times 8$. The fits using Eq. (2.3) are plotted. 2-loop result for the trace anomaly density in HTL perturbation theory, from [28], is shown as a shaded band that correspond to varying μ by a factor of two around $2\pi T$.
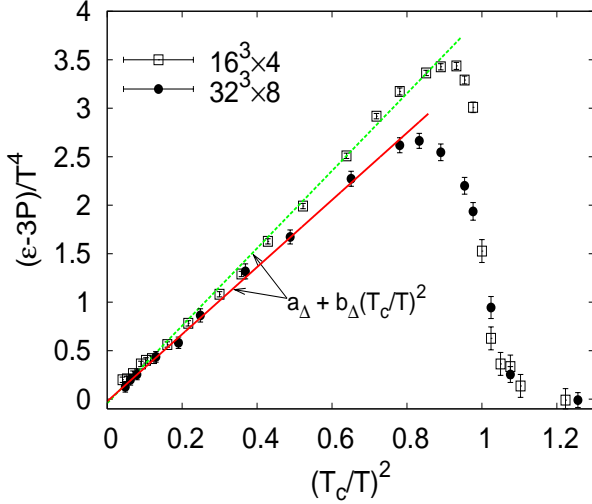

FIG. 2: The trace anomaly density $(\epsilon - 3P)/T^4$ as a function of $1/T^2$ (in units of T_c). Same data as in Fig. 1.

accounted for in perturbation theory. It should be mentioned that similar power corrections have been identified in the trace anomaly in gluodynamics in $2+1$ dimensions [27].

Integrating the trace anomaly, as modeled in (2.3), from $T = T_c$ and using the continuity of the pressure across the phase transition yields the following form for the pressure

$$\frac{P(T)}{T^4} = \frac{b_\Delta}{2}\left(1 - \frac{T_c^2}{T^2}\right) + a_\Delta \log\left(\frac{T}{T_c}\right) + \frac{P(T_c)}{T_c^4}. \tag{2.6}$$

Clearly this expression cannot be used at very high temperatures unless $a_\Delta = 0$, a result suggested by the fit. On the opposite extreme, integrating from very high energies, where radiative corrections should dominate, yields

$$\frac{P(T)}{T^4} = \frac{b_0}{2} - \int_T^\infty \frac{dT'}{T'}\Delta_{\text{pert}}(T') \tag{2.7}$$

where

$$b_0 = (N_c^2 - 1)2\pi^2/45 \approx 3.51. \tag{2.8}$$

Matching at some temperature $T = T_p$, gives b_Δ from $P_c = P(T_c)$, b_0 and $\Delta_{\text{pert}}(T)$ (assuming $a_\Delta = 0$). Clearly, since $P_c > 0$, and $\Delta_{\text{pert}}(T) > 0$, we expect $b_\Delta < b_0$. (Note that the power correction should be negligible at the matching temperature.) The result from the fit almost saturates the inequality since the continuum limit extrapolation yields $b_\Delta = 3.28(27)$ which suggests that *both* P_c and $\Delta_{\text{pert}}(T)$ must be very small. The analysis of Section III confirms these issues.

III. THERMODYNAMICS OF THE PHASE TRANSITION

The analysis of the previous section leaves little doubt on the existence of power corrections in gluodynamics down *almost* to the critical temperature. Moreover, at the maximum temperature measured on the lattice, $T = 5\,T_c$, the power contribution still dominates the result, and both terms become comparable only at higher temperatures, $T \sim 10\,T_c$. One sensible question to ask is what is the temperature where perturbation theory applies. We address this question in more detail in Section IV.

If for the moment perturbative corrections are disregarded (they are small anyhow), it is worth reviewing why the previous results are completely against the idea of a gluon plasma. Actually, the standard textbook argument requires the bag pressure to equilibrate the free gluon gas at the critical temperature

$$P(T) = P_{\text{gluons}}(T) - B, \qquad T > T_c, \tag{3.1}$$

yielding the trace anomaly

$$\Delta = \frac{4B}{T^4}, \qquad T > T_c. \tag{3.2}$$

As already shown, this behaviour is excluded by lattice data.

In this section we elaborate on a suggestion by Pisarski [10],

$$P(T) = \frac{b_0}{2}(T^4 - T_c^2 T^2), \tag{3.3}$$

which achieves a fair description of the data assuming infinitely heavy glueball masses, negligible radiative perturbative corrections, and a temperature dependent bag

energy, called *fuzzy bag*,

$$B_{\text{fuzzy}} = \frac{b_0}{2} T_c^2 T^2. \qquad (3.4)$$

As shown in Section V, dimension-2 condensates provide a natural explanation of this fuzzy bag picture *including* the correct power behaviour and coefficient, while simultaneously account for power corrections in the Polyakov loop and the $q\bar{q}$-potential at a quantitative level. A study of the thermodynamics of this fuzzy constant is also made in Ref. [29] within the holographic QCD approach based on the AdS/CFT correspondence.

A. Infinitely massive glueballs

At very high temperatures, where radiative corrections become negligible, the free gluonic gas relation $P(T) \sim (N_c^2 - 1)\pi^2 T^4/45$ holds. On the other hand, in the confined phase, i.e., below the critical temperature T_c, the spectrum is believed to be saturated by glueballs (for reviews see e.g. Ref. [30, 31]). Since the pressure must be a continuous function across the phase transition, one finds

$$P(T_c) = P_{\text{glueballs}}(T_c). \qquad (3.5)$$

However, the lightest glueball has $J^{PC} = 0^{++}$ and $M_{0^{++}} = 1.73(1)$GeV. This is much heavier than the critical temperature $T_c \approx 270$ MeV (a mysteriously disparate scale [32]). Thus Δ is very small (though not exactly zero) in the confined phase. Taking the infinite glueball mass limit produces $P_{\text{glueballs}} = 0$. These conditions are met by the interpolating functions of the form

$$P = \frac{(N_c^2 - 1)\pi^2}{45} T^4 \left[1 - \left(\frac{T_c}{T} \right)^n \right], \qquad T \geq T_c, \ (3.6)$$

with arbitrary positive n. This parameterization yields

$$\Delta = \frac{n(N_c^2 - 1)\pi^2}{45} \left(\frac{T_c}{T} \right)^n, \qquad T \geq T_c, \qquad (3.7)$$

which for $n = 2$ corresponds to take $a_\Delta = 0$ and $b_\Delta = 3.51$ in Eq. (2.3), in excellent agreement with the fit to the lattice data of the previous section. This thermodynamic consistency does not explain however *why* there is a power correction with $n = 2$. A fit of Eq. (3.7) to the data of Fig. 2 for the same range as in Eq. (2.5), $1.13 T_c \leq T \leq 4.54 T_c$, yields $n = 1.97(5)$ with a slightly larger $\chi^2/\text{dof} = 0.62$.

B. Finite mass glueball spectrum

Given the goodness of the description we may try to improve by actually computing the pressure of an ensemble of finite mass glueballs in the confined phase using the 12 glueball spectrum currently determined on the lattice [33, 34]. Quite generally such a calculation would require the use of the quantum field theoretical version of the virial expansion [35] and thus to take into account corrections from binary, ternary and higher order collisions. For instance, for a binary collision the threshold for the first possible lowest glueball-glueball resonance is located at twice the glueball mass $2M_{0^{++}} \sim 3$GeV which lies above the 5th glueball and is expected to produce an insignificant contribution for $T < T_c \approx 270$ MeV. Fortunately, the first four low-lying glueballs computed on the lattice are kinematically stable bound states. Actually, within a large N_c framework one has a gas of stable and non-interacting glueballs since $M_{\text{gb}} \sim N_c^0$ and $V_{\text{gb-gb}} \sim 1/N_c^2$. So it seems safe to treat the glueballs as a gas of free bosons. This yields

$$P_{\text{glueballs}}(T) = \frac{1}{3} \sum_i g_i \int \frac{d^3 k}{(2\pi)^3} \frac{\vec{k} \cdot \nabla_k E_k}{e^{E_k/T} - 1}, \ (3.8)$$

where the index i runs on the glueball species: M_i denotes the glueball mass and $g_i = 2J_i + 1$ the corresponding angular momentum degeneracy. A convenient low temperature expansion gives

$$P_{\text{glueballs}}(T) = \sum_i \sum_{n=1}^{\infty} \frac{g_i M_i^2 T^2}{2n^2\pi^2} K_2 \left(\frac{M_i n}{T} \right) \qquad (3.9)$$

where the sum over n corresponds to the thermal loops and K_2 is a modified Bessel function of the second kind.

Using known spectrum of the lightest 12 glueball species [33, 34] one can evaluate $P_c = P_{\text{glueball}}(T_c)$. This requires fixing the critical temperature scale T_c where the trace anomaly is evaluated [9] on the one hand and the Sommer scale r_0 defined as $r_0^2 V'_{q\bar{q}}(r_0) = 1.65$ [36], where the glueball spectrum is obtained [33, 34], on the other. This is necessary since the numerical value of P_c is quite sensitive to the glueball mass to critical temperature ratio, M_i/T_c.

Boyd *et al.* [9] find $T_c/\sqrt{\sigma} = 0.629(3)$ while Luscher *et al.* [37] obtain $r_0\sqrt{\sigma} = 1.22(8)$. This is consistent with the elaboration of Teper [30] where $r_0\sqrt{\sigma} = 1.195(10)$, $T_c/\sqrt{\sigma} = 0.640(15)$ and $M_{0^{++}}/\sqrt{\sigma} = 3.52(11)$ are quoted as well as the fixed point action of Niedemeyer *et al.* [38] simulation where the values $r_0\sqrt{\sigma} = 1.197(11)$, $T_c/\sqrt{\sigma} = 0.624(7)$, $r_0 T_c = 0.750(5)$ and scalar $r_0 M_{0^{++}} = 4.12(21)$ and tensor $r_0 M_{2^{++}} = 5.96(24)$ low-lying glueball masses. Along similar lines Necco [39], gets $r_0 T_c = 0.7498(50)$. These latter values agree within errors with the 12 glueball spectrum simulation [33, 34] where $r_0 M_{0^{++}} = 4.21(11)$ and $r_0 M_{2^{++}} = 5.85(2)$ vs $r_0 M_{0^{++}} = 4.16(11)$ and $r_0 M_{2^{++}} = 5.85(5)$ are obtained respectively. Using for definiteness $r_0 T_c = 0.75$ we get the result

$$P_c = P_{\text{glueball}}(T_c) = 0.01(1) \qquad (3.10)$$

where the error has been estimated from the uncertainty in the glueball masses. This value of the pressure is consistent with the fit to the data, although the glueball

spectrum contribution is not so strongly constrained by the trace anomaly data. About 80% of the total contribution to a_Δ is given by the lightest scalar 0^{++} and tensor 2^{++} glueball states, confirming the marginal role of the excited glueball spectrum as well as possible two-glueball resonances.

The trace anomaly due to the glueballs reads

$$\Delta(T) = \sum_i \sum_{n=1}^\infty \frac{g_i}{2n\pi^2} \frac{M_i^3}{T^3} K_1\left(\frac{M_i n}{T}\right). \qquad (3.11)$$

At the lowest lattice temperature $T = 0.89\,T_c$ one has $\Delta < 0.08$. This sets an upper bound since all contributions in Eq. (3.11) are positive. Taking the known glueball spectrum one gets $\Delta_{\rm gb}(0.89 T_c) = 0.03(6)$. However, the steep raise below T_c is not reproduced by the glueball spectrum, basically due to the heavy values of the masses. In this context it has been suggested [40] that above the phase transition glueballs lower their masses providing an enhancement already below the phase transition. The behaviour and effect of glueballs in the deconfined phase has also been considered in [41]. Note also that the glueball gas formula neglects glueball interactions which are nominally $1/N_c^2$. Assuming that the multiplicity in the glueball spectrum is $\mathcal{O}(N_c^0)$ this yields a trace anomaly $\Delta = \mathcal{O}(N_c^0)$ for $T < T_c$, while we clearly have $\Delta = \mathcal{O}(N_c^2)$ for $T > T_c$.

Further considerations on the glueball spectrum and its possible Hagedorn structure are presented in Appendix B. In all, our analysis above explains why the pressure of finite mass glueballs is compatible with zero within errors both from the trace anomaly lattice calculation as well as from the direct glueball spectrum estimate, and hence supports the infinite mass glueball assumption underlying the model of Ref. [10].

IV. RENORMALIZATION GROUP IMPROVEMENT OF THE PERTURBATIVE FREE ENERGY

The weak coupling expansion for the free energy of the quark-gluon plasma has been calculated through order $\alpha_s^{5/2}$ [42, 43, 44], and the result yields poor convergence at the lattice QCD available temperatures $T < 5T_c$, and even at much higher temperatures. The problem is more involved taking into account that at order $\alpha_s^3 \log \alpha_s$ starts the contribution of non perturbative effects which is hard to compute [45]. There have also been numerous attempts to resum perturbation theory in order to get a better convergence of the result (see e.g. [46] and references therein). One of the most developed techniques is the Hard Thermal Loop (HTL) perturbation theory, first proposed in [47, 48]. This is an efficient reorganization of the perturbative series which avoids the unpleasant infrared singularities. We show in Fig. 1 the two-loop result for the trace anomaly density from HTL [28] (see also [49] for a one loop computation). The HTL result becomes

a very smooth function in the regime $T > 1.13\,T_c$ and is unable to reproduce the lattice data. One of the most conspicuous facts is that the pressure from HTL remains nearly constant with a value of about 95% of that of an ideal gas of gluons, even for temperatures $\sim 10^3\,T_c$. This means that the approach to the ideal gas is extremely slow, which ultimately implies that the trace anomaly is very small and smooth. The same feature is shared by the weak coupling expansion and other resummation techniques, and it seems to be confirmed by preliminary results of lattice computations at extremely high temperatures [50].

Many perturbative studies based on the computation of the three-dimensional effective theory of QCD on the lattice lead once again to the same smooth behaviour for the trace anomaly even near T_c [51, 52].

These problems can be understood within our framework by noting that perturbation theory, as well as HTL and other resummation techniques, contain only logarithms in the temperature, suggesting a mild temperature dependence. Our discussion above shows that these approaches would yield a powerless contribution, $\Delta_{\rm pert}$, which should ultimately be identified with the *almost* constant and vanishing a_Δ of Eq. (2.3) rather than with the full result from the lattice. Actually, the maximum lattice temperature $T = 5\,T_c$ should be far from the pQCD estimate since the power correction provides the bulk of the full result at this temperature. For $N_c = 3$ the $\mathcal{O}(g^5)$ corrections to Eq. (A12) corresponds to multiply it by $(1 - 6g(\mu)/\pi)$ [53] which becomes small for $g(\mu) \ll \pi/6$ or $\mu \gg 10^{11}\Lambda_{\rm QCD}$. This delayed onset of perturbative QCD is not a new phenomenon and takes place in the study of exclusive processes at high energies where there appear collinear divergences (see e.g. Ref. [54] and references therein). In order to support this statement, we zoom in Fig. 3 the two-loop result for the trace anomaly density in HTL and compare it with our value $a_\Delta = -0.02(9)$ from a fit of lattice data using Eq. (2.3), estimated in the continuum limit. The HTL result is inside the error band for temperatures $T > 1.4(3)\,T_c$. From a fit of $N_\tau = 8$ lattice data we got $a_\Delta = -0.02(4)$, and the HTL result is inside the error band for $T > 1.7(4)\,T_c$. It is appropriate to mention at this point that the estimated error of the 2-loop HTL start to be important below $1.5\,T_c$ as we can see in Fig. 3. Moreover below $1.5\,T_c$ the 1-loop and 2-loop computations in HTL start to be in disagreement [28].

There are in the literature other computations of the equation of state of gluodynamics from lattice simulations that are in qualitative agreement with Boyd et al. (see Ref. [18, 19, 20, 21] and references therein). Power corrections have also been obtained in the equation of state of full QCD [55]. Nonetheless we will focus in this paper only on pure gluodynamics for simplicity.

The conclusion of the above discussion is that the behaviour of Eq. (2.3) clearly contradicts perturbation theory which contains no powers but only logarithms in the temperature. Power corrections are understood as the
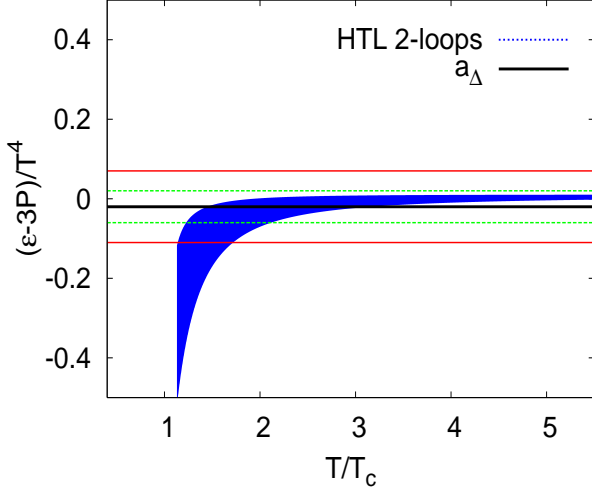
FIG. 3: The trace anomaly density $(\epsilon - 3P)/T^4$ as a function of T (in units of T_c). The 2-loop HTL result from ref. [28] is shown as a shaded band that corresponds to varying the scale μ by a factor of two around $2\pi T$. The horizontal lines display the fitted value of a_Δ in Eq. (2.3), from $N_\tau = 8$ lattice data, $a_\Delta = -0.02 \pm 0.04$ (error bar in green), and the continuum limit estimate, $a_\Delta = -0.02 \pm 0.09$ (error bar in red).

high energy trace of non-perturbative low energy effects. This kind of corrections have already been detected and analyzed in the Polyakov loop and in the heavy quark-antiquark free energy in terms of dimension two gluon condensates [4, 5]. The rather successful description of the lattice data for the trace anomaly suggests a similar approach for the equation of state, that will be analyzed in Sec. V.

Our previous numerical analysis of lattice data suggests that the perturbative series of the free energy of a hot gluon plasma is rather small, not only in the high temperature limit, but also near the deconfined phase. The perturbative series has a problem of convergence for temperatures below $10\,T_c$, which could mean a lack of analyticity near the phase transition.

We can improve the convergence by considering a re-summation of the perturbative series based on renormalization group invariance. The details of the computation are provided in Appendix C. In practice this means to reorganize the perturbative series and express it in terms of the following manifestly Renormalization Group (RG) invariant

$$\tilde{\alpha}(T) = \frac{1}{\gamma_0} \frac{1}{\log\left(\frac{2\pi T}{\Lambda_{\text{QCD}}}\right)}. \qquad (4.1)$$

The result for the pressure so obtained is a manifestly RG invariant series up to order $\mathcal{O}(\tilde{\alpha}^3)$ and is given by Eq. (C10) with coefficients Eq. (C11). The only unknown coefficient to this order is A_6, which enters at $\mathcal{O}(\tilde{\alpha}^3)$. The trace anomaly involves a derivative of the pressure with respect to temperature, see Eq. (2.2). Using (4.1), this
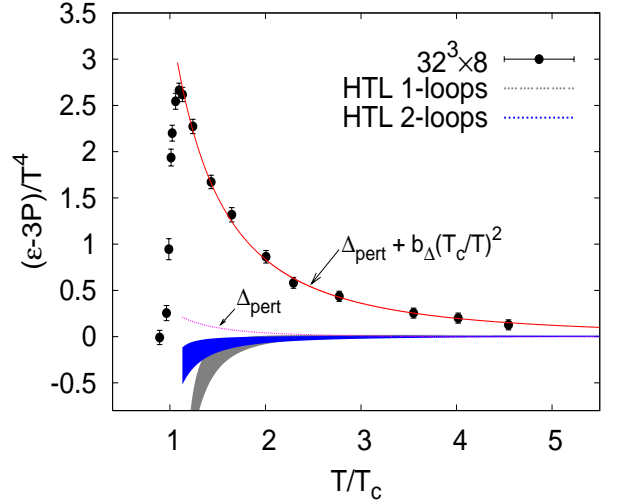

FIG. 4: The trace anomaly density $(\epsilon - 3P)/T^4$ as a function of T (in units of T_c). Lattice data are from [9] for $N_\sigma^3 \times N_\tau = 32^3 \times 8$. The fit using Eq. (4.3) is plotted. We also plot the perturbative contribution from Eq. (4.2), and compare it with the 1-loop [49] and 2-loop [28] HTL results (shaded bands).

can be computed as

$$\Delta_{\text{pert}} = -\frac{(N_c^2 - 1)\pi^2}{45}\gamma_0\tilde{\alpha}^2\frac{dH_{\text{pert}}}{d\tilde{\alpha}}, \qquad (4.2)$$

where we have defined $H_{\text{pert}} = P_{\text{pert}}/P_{\text{ideal}}$, being $P_{\text{ideal}} = ((N_c^2 - 1)\pi^2/45)T^4$ the pressure of an ideal gas of massless gluons. This result holds modulo $\mathcal{O}(\tilde{\alpha}^{9/2})$ when Eq. (C10) is considered.

One can try to compare the RG-invariant Δ_{pert}, truncated to $\mathcal{O}(\tilde{\alpha}^{7/2})$ (included) with the available lattice data. In all cases we use the more accurate $N_\tau = 8$ data, including points in the range $1.13 \leq T/T_c \leq 4.5$. To begin with we allow both A_6 and Λ_{QCD} to change as free parameters. In this form no acceptable fit is achieved. The best fit gives $\chi^2/\text{dof} = 1.6$ with $A_6 = 0.33 \pm 3.0$. It mimics the overall shape of the data but fails to reproduce the highest temperature data, where perturbation theory (PT) is expected to be more reliable. In addition, this best fit requires unphysical values of $\Lambda_{\text{QCD}} \sim 338\,\text{MeV}$, which would not reproduce other gluodynamics data. To avoid this problem, in what follows we fix Λ_{QCD} to lie on the range $0.877(88)\,T_c$, allowing for a 10% uncertainty in this parameter. In this case, fitting A_6 in order to reproduce the highest temperature data, one can verify that the χ^2/dof rapidly deteriorates as new lower temperature data are being included in the fit, while simultaneously the fitted value of A_6 rapidly changes to larger negative values.

Next we proceed to add the non perturbative (NP) term $b_\Delta(T_c/T)^2$ to the perturbative ones and try to reproduce the data using A_6 and b_Δ as parameters. In a more detailed treatment one should expect some interference between these contributions, from radiative corrections in the form of anomalous dimension, etc, in the
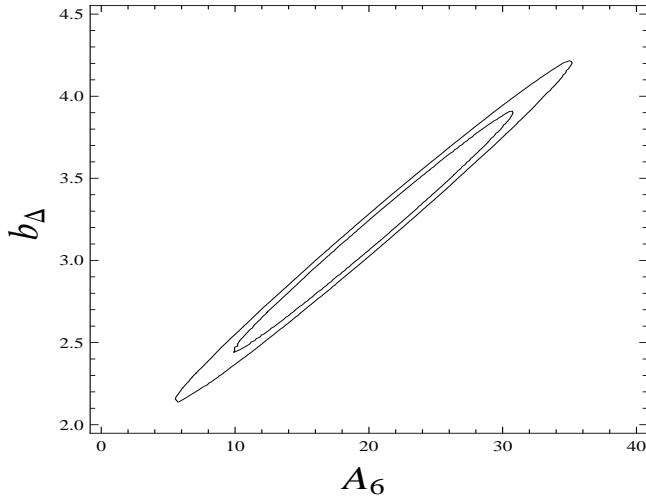
FIG. 5: Correlation ellipses corresponding to $\Delta\chi^2 = 2.3, 4.7$ (dof = 6) between the perturbative parameter A_6 and the non perturbative parameter b_Δ (see main text) characterizing the trace anomaly. The central values are obtained using the lattice data in the interval $1.13 T_c \leq T \leq 4.54 T_c$.
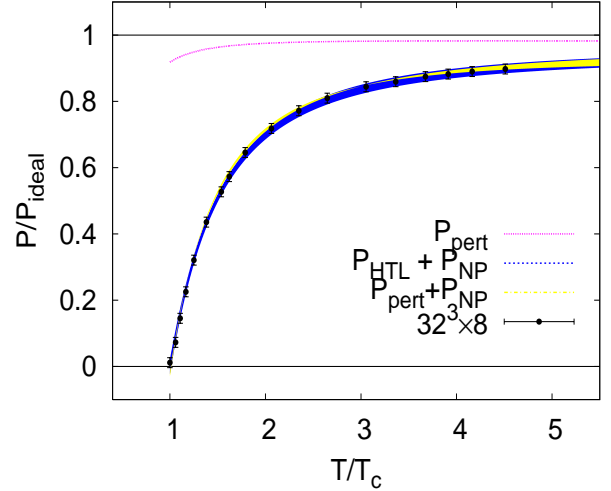

FIG. 6: The pressure P as a function of T (in units of T_c). Lattice data are from [9] for $N_\sigma^3 \times N_\tau = 32^3 \times 8$. We plot the fit of Eq. (4.4), and the analogous equation with P_{HTL} to 2-loops [28]. We retain the same values of b_Δ as in Fig. 4. The perturbative contribution with no extra contributions is also displayed.

NP term. Presently, we adopt the simplest scenario of additive PT and NP contributions, i.e. we consider

$$\Delta = \Delta_{\text{pert}} + b_\Delta \left(\frac{T_c}{T}\right)^2 . \qquad (4.3)$$

This procedure yields a fairly good fit to the data, $\chi^2/\text{dof} = 0.40$, with $b_\Delta = 3.18(74)$ and $A_6 = 20.0 \pm 10.5$ (see Fig. 4). This fit is only slightly worse than that in section II, which used a constant perturbative background a_Δ. The error bars in b_Δ and A_6 are enhanced as they include the uncertainty in Λ_{QCD}. It is also noteworthy that the two parameters are highly correlated (see Fig. 5), and in fact the combination $A_6 - 14.2\, b_\Delta = -25.1 \pm 1.3$ (which has zero correlation with b_Δ) has a much smaller error. Nevertheless, we caution that these numbers are to be taken as indicative only. The missing terms in the perturbative expansion make the perturbative contribution to be little reliable at temperatures near the transition. For instance, if the previous fit is repeated using data in the range $1.43 \leq T/T_c \leq 4.5$, the central value of b_Δ increases to 3.47 while A_6 changes correspondingly.

A remarkable fact is that the central value of A_6 is such the Δ_{pert} falls on top of the 2-loop HTL calculation. Actually, the value of A_6 required to produce this match between the two perturbative calculations at $T = 4.5\, T_c$ is $A_6 = 19.0 \pm 1.2$ (note that this latter number does not depend on lattice trace anomaly data). Similar values, $A_6 = 20.0 \pm 0.8$, allow to reproduce the HTL result by Δ_{pert} in the range $2.75 \leq T/T_c \leq 4.5$ with $\chi^2/\text{dof} = 0.67$ (where we include the uncertainty from Λ_{QCD} and from the scale μ in HTL). While the central values of A_6 obtained from a fit to lattice data, including the NP

term, and to 2-loop HTL are equal (20.0), the uncertainty assigned from the second determination is much smaller. (Nevertheless, the preferred value of A_6 as extracted from a match to HTL evolves at very high temperatures to a value close to zero.)

We have also considered a fit to the lattice trace anomaly data using the 2-loop HTL result as perturbative background, plus the non perturbative term, in the range $1.24 \leq T/T_c \leq 4.5$. This gives $b = 3.62(11)$ with $\chi^2/\text{dof} = 0.72$. So it also provides a good fit of the data, only slightly worse than that from the RG invariant Δ_{pert}.

In all cases the perturbative contribution is subdominant and very small above $2 T_c$. Also, the pressure of gluodynamics is very well reproduced for all temperatures above T_c, after inclusion of the non perturbative term. Integrating the trace anomaly, as modeled in (4.3), yields the following form

$$\frac{P(T)}{T^4} = c_p + \frac{P_{\text{pert}}(T)}{T^4} - \frac{b_\Delta}{2} \left(\frac{T_c}{T}\right)^2 , \qquad (4.4)$$

where c_p is an additive constant related to the normalization of the thermodynamic quantities on the lattice. Note that $P_{\text{pert}}(T)$ is known from Eq (C10). A fit of the lattice data for the pressure using Eq. (4.4) is plotted in Fig. 6. We retain in the fit the same value of b_Δ that we previously obtained from the trace anomaly, and consider as free parameters A_6 and c_p. This formula yields a good fit to the data in the regime $1 \leq T/T_c \leq 4.5$ with $A_6 = 20 \pm 10$, $c_p = -0.08 \pm 0.08$ and $\chi^2/\text{dof} = 0.26$. As expected, the value of A_6 agrees with our previous determination from the trace anomaly. We also consider

a fit of the lattice data using the 2-loops HTL results for the pressure, P_{HTL} [28]. The only free parameter in this case is c_p, and we get $c_p = 0.006 \pm 0.020$, $\chi^2/\text{dof} = 0.96$. We have increased the errors for the pressure of Ref. [9] by a factor 2.5, and it seems to be reasonable taking into account the difference between lattices with different temporal extent N_τ.

We have also analyzed different lattice data other than those primarily considered above [9], namely Ref. [18] which almost reproduces [9] and Ref. [20] which differ from those of Ref. [9] in the higher temperature region, possibly due to systematic errors not provided in Ref. [20]. An estimate of the systematic errors based on a variation of lattice sizes suggests that they could be of the order of 0.3 in Δ, which explains the discrepancy of 0.15 between both lattice data. Thus, we have artificially increased the errors of Ref. [20] by a factor of ten. For the resummed PT+NP approach we get $A_6 = 31(7)$ and $b_\Delta = 3.9(0.5)$ with $\chi^2/\text{dof} = 0.40$ for the data of Ref. [18] while $A_6 = -9(18)$ and $b_\Delta = 1.42(1.45)$ with $\chi^2/\text{dof} = 0.12$ when data from Ref. [20] are used. Note that the resulting pairs (A_6, b_Δ) fall reasonably close to the line $A_6 - 14.2 \, b_\Delta = -25.1 \pm 1.3$. The HTL+NP yields $b_\Delta = 3.44(0.09)$ with $\chi^2/\text{dof} = 2.6$ for Ref. [18], and $b_\Delta = 4.05(0.15)$ with $\chi^2/\text{dof} = 0.44$ for Ref. [20]. With the provisos spelled out previously, it is quite reasonable to conclude that other lattice data sets confirm our results, with larger uncertainties.

To finish this section we note that the trace anomaly as obtained from Δ_{pert} or from HTL does not have a definite sign, but this quantity becomes positive definite for all temperatures above T_c upon inclusion of the non perturbative term $b_\Delta (T_c/T)^2$, for values of b_Δ fitting the data. Curiously, the fitted value $A_6 = 20$ is roughly the threshold value above which Δ would fail to be positive definite (the region with negative values being located around $T = 50 \, T_c$).

V. DIMENSION TWO CONDENSATE AND TRACE ANOMALY

We can make a first approximation to the computation of the trace anomaly using the non perturbative model introduced in [4, 6]. There, the inverse temperature power corrections observed in lattice data for the Polyakov loop [63] and the $q\bar{q}$-potential [64], are accounted for by means of a non perturbative contribution to the zeroth component gluon propagator of the dimensionally reduced theory. Namely,

$$D_{00}(\mathbf{k}) = D_{00}^{\text{P}}(\mathbf{k}) + D_{00}^{\text{NP}}(\mathbf{k}), \qquad (5.1)$$

where

$$D_{00}^{\text{P}}(\mathbf{k}) = \frac{1}{\mathbf{k}^2 + m_D^2}, \quad D_{00}^{\text{NP}}(\mathbf{k}) = \frac{m_G^2}{(\mathbf{k}^2 + m_D^2)^2}. \quad (5.2)$$

Here m_D is the Debye mass, which scales like T, and m_G is a temperature independent mass parameter. The

non perturbative term in the gluon propagator induces a corresponding contribution to the dimension two gluon condensate:

$$\langle A_{0,a}^2 \rangle^{\text{NP}} = \frac{(N_c^2 - 1)T m_G^2}{8\pi m_D}. \qquad (5.3)$$

Being the trace anomaly in gluodynamics given by Eq. (2.1), the problem can be addressed through the computation of the vacuum expectation value of the squared field strength tensor $\langle (G_{\mu\nu}^a)^2 \rangle$.

In perturbation theory $\langle G_{\mu\nu,a}^2 \rangle$ starts at $\mathcal{O}(g^2)$, cf. (A12). This comes from integration of "hard modes", i.e., modes with momentum scale $2\pi T$ [65]. The contribution of soft modes, described by the dimensionally reduced Lagrangian, starts at $\mathcal{O}(g^3)$. These perturbative contributions break scale invariance through radiative corrections and so depend logarithmically on the temperature, after extracting the canonical factor T^4. Power corrections require a stronger breaking of scale invariance. We will assume that this does not happen for the hard modes since condensates are low energy phenomena. Therefore, we assume that the non perturbative scale breaking enters through the parameter m_G^2 in (5.2) corresponding to the temporal gluon propagator. Just by dimensional counting, this parameter will produce the required power correction, $\sim 1/T^2$ in the observable Δ. The ultraviolet divergent momentum integrals in three dimensions are dealt with by dimensional regularization. On the other hand no non perturbative scale breaking is assumed for spatial gluons. As these gluons fail to have a Debye mass, a non perturbative term of the type $m_G'^2/k^4$ in the propagator of spatial gluons would yield a vanishing contribution within dimensional regularization. This approach yields

$$
\begin{aligned}
\langle G_{\mu\nu,a}^2 \rangle^{\text{NP}} &= 2\langle (\partial_i A_{0,a})^2 \rangle^{\text{NP}} \\
&= 2T(N_c^2 - 1) \int \frac{d^3k}{(2\pi)^3} k^2 D_{00}^{\text{NP}} \quad (5.4) \\
&= -T(N_c^2 - 1)\frac{3}{4\pi} m_D m_G^2.
\end{aligned}
$$

The negative sign is a consequence of the renormalization.[1] Making use of (5.3), this result can be expressed in a more conveniently form in terms of the dimension two gluon condensate, and finally gives

$$(\epsilon - 3P)_{\text{NP}} = -3\frac{\beta(g)}{g} m_D^2 \langle A_{0,a}^2 \rangle^{\text{NP}}. \qquad (5.5)$$

The behaviour is $\sim T^2$, in contrast to the perturbative behaviour $\sim T^4$. It is also noteworthy that $(\epsilon - 3P)_{\text{NP}}$ satisfies the correct large N_c counting, being $\mathcal{O}(N_c^2)$.

---

[1] As advertised, an $\mathcal{O}(g^3)$ contribution, namely, proportional to m_D^3, is derived from the analogous computation using the *perturbative* component of the gluon propagator.

The coefficient b_Δ in the parameterization (2.3) is to be identified with the estimate of Eq. (5.5)[2]

$$b_\Delta T_c^2 = -3\frac{\beta(g)}{g}\frac{m_D^2}{T^2}\langle A_{0,a}^2\rangle^{\rm NP}.\qquad(5.6)$$

Therefore, since $\beta(g) < 0$, we have a positive b_Δ. If we consider the perturbative value of the beta function $\beta(g) \sim g^3 + \mathcal{O}(g^5)$, the r.h.s. of Eq. (5.6) shows a factor g^2 in addition to the dimension two gluon condensate $g^2\langle A_{0,a}^2\rangle^{\rm NP}$. So the fit of the trace anomaly data is sensitive to the value of g. Fortunately, this quantity has only a smooth dependence on T. For the Polyakov loop the sensitivity in g is only through the perturbative terms, which are much smaller than the NP ones. When we consider the perturbative value g_P to 2-loop, $g_P \sim 1.41$, we get from the fit of the trace anomaly

$$g^2\langle A_{0,a}^2\rangle^{\rm NP} = (0.72(3)~{\rm GeV})^2.\qquad(5.7)$$

Here we have used the continuum limit estimate $b_\Delta = 3.28(27)$ of section II, based on a constant perturbative background. For the present purposes we consider this estimate sufficiently accurate and less subject to details related to the precise determination of the perturbative background. The precise number for b_Δ varies by using any of the determinations of b_Δ discussed in section IV but they are still compatible with each other.

The value quoted in Eq. (5.7) is a factor 1.5 smaller than the value obtained from a fit of the Polyakov loop $(0.84(6)~{\rm GeV})^2$ [4] and heavy $q\overline{q}$ singlet free energy $(0.90(5)~{\rm GeV})^2$ [6]. This disagreement could be explained in part on the basis of certain ambiguity of g in the non perturbative regime. We show in Ref. [6] that a better fit of the Polyakov loop and heavy quark free energy lattice data in the regime $T_c < T < 4\,T_c$ is obtained for a value of g slightly smaller than g_P, i.e. $g = 1.26 - 1.41$. Taking this, we get

$$\begin{aligned}g^2\langle A_{0,a}^2\rangle^{\rm NP} &= (0.77(6)~{\rm GeV})^2\\&= (2.84 \pm 0.21\,T_c)^2,\qquad(5.8)\end{aligned}$$

in better agreement with determinations of the condensate from previous observables, see Table I. This value follows from the continuum limit estimate of b_Δ, and its discrepancy with the corresponding value from a fit of $N_\tau = 8$ lattice data is 5%, which is much smaller than the error. The same happens with the lattice data of the Polyakov loop [4].

---

[2] It may look unnatural that the rather complicated r.h.s. of this formula yields b_Δ, when Pisarski's argument, section III, assigned to this quantity an almost geometrical value, $(N_c^2 - 1)2\pi^2/45$. However, this is not so. In the present model, the geometrical value of b_Δ together with the r.h.s. provide rather an estimate of T_c, identified as the temperature where the pressure becomes negligible.

Observable	$g^2\langle A_{0,a}^2\rangle^{\rm NP}$
Polyakov loop [4]	$(3.11 \pm 0.22\,T_c)^2$
Heavy $\overline{q}q$ free energy [6]	$(3.33 \pm 0.19\,T_c)^2$
Trace Anomaly	$(2.84 \pm 0.21\,T_c)^2$

TABLE I: Values of the dimension two gluon condensate from a fit of several observables in the deconfined phase of gluodynamics: Polyakov loop, singlet free energy of heavy quark-antiquark and trace anomaly. Values are in units of T_c. We show the fit for lattice data with $N_\tau = 8$ for the heavy $\overline{q}q$ free energy, and the continuum limit estimate for the others. Error in last line takes into account an indeterminate value of the coupling constant $g = 1.26 - 1.41$, being the highest value the perturbative g_P up to 2-loop at $T = 2\,T_c$. The critical temperature in gluodynamics is taken as $T_c = 270 \pm 2\,{\rm MeV}$ [66].

In a recent paper the electric-magnetic asymmetry of the condensate defined as

$$\Delta_{A^2}(T) = g^2\langle A_{0,a}^2\rangle - \frac{1}{3}g^2\langle A_{i,a}^2\rangle,\qquad(5.9)$$

is computed in SU(2) Yang-Mills theory [67]. Obviously this quantity vanishes at zero temperature by Euclidean symmetry. The dimension two condensate has dimension of mass squared and so it would vanish in a perturbative calculation at zero temperature. At finite temperature instead $\langle A_{0,a}^2\rangle^{\rm P}$ scales as T^2 modulo slowly varying radiative corrections. On the other hand our model assumes that the non perturbative part of the condensate $\langle A_{0,a}^2\rangle^{\rm NP}$ is temperature independent (modulo radiative corrections) at least in the regime slightly above the phase transition and beyond This is exactly the behaviour that is obtained in [67] for the asymmetry in this regime. If we naively scale our determination for $g^2\langle A_{0,a}^2\rangle^{\rm NP}$ with the number of gluons 3/8, we get in SU(2) the estimate $g^2\langle A_{0,a}^2\rangle^{\rm NP} = (1.9(2)T_c)^2$. This value can be compared with the temperature independent contribution to (5.9), $\Delta_{A^2}^{(0)} = (0.894(14)\,T_c)^2$ of [67], leaving some room for a magnetic contribution in our model. In any case, the very existence of the dimension two condensate seems to be confirmed by the work of ref. [67].

VI. CONCLUSIONS

For quite a long time it has been believed that the highest temperature measured in lattice calculations $T_{\rm max} \sim 5\,T_c$ could be matched smoothly to perturbative calculations. Assigning the standard $\overline{\rm MS}$ conversion factor to a momentum scale $\mu \sim 2\pi T$ and taking $T_c \sim 0.6\sqrt{\sigma}$ yields $\mu_{\rm max} \sim 1.5\,{\rm GeV}$; a high scale. Actually, there have been many perturbative studies that unsuccessfully tried to reproduce the lattice data for the trace anomaly. As we have shown in this work, the trace anomaly density shows unmistakable traces of power corrections in the inverse temperature in the region slightly above the

phase transition which may be accommodated by dimension two gluon condensate. This finding parallels a similar analysis for the Polyakov loop and the values for the dimension two condensate is in satisfactory agreement with the value found here. The 20% discrepancy may be due to many different sources such as anomalous dimension effects. It is important to emphasize that the non-triviality of such a numerical consistency can only be addressed in a calculation where all quantities can be estimated simultaneously. In addition, it would be worrisome if the dimension two condensate associated with the power correction would differ by an order of magnitude. The fact that this is by far not the case suggests searching for other features where the condensate might generate further genuinely non-perturbative thermal power corrections. This of course applies not only to gluodynamics but also to the case of full QCD with active dynamical quarks and in the presence of a chemical potential. A possible guideline for future studies would be a simple minded extension of the power corrections ideas discussed in the present work. Actually, most recent QCD calculations already confirm the ubiquitous $1/T^2$ behaviour of the trace anomaly density [55].

On a more methodological level, let us remind that at high energies and zero temperature momentum power corrections are widely accepted and encode via condensates non-perturbative effects. For this situation the full machinery of QCD sum rules has been extensively developed and applied with recurrent success. While there is no reason why condensates should not be present at high temperatures as genuinely non-perturbative effects even after deconfinement, temperature power corrections above the phase transition are much less common and a more complete technology than that used in the present work would be most useful. In addition to the standard finite temperature complications one must also face the well known ambiguity on clear separation between perturbative effects from existing non-perturbative features. There is no doubt that this intricate theoretical problem should necessarily be addressed at some stage in the near future.

In spite of the overall numerical consistency found for the dimension two condensate, the physics underlying the generation of such a condensate, and ultimately the very appearance of thermal power corrections remains uncertain. Actually, the elusiveness of a theoretical determination of the dimension two condensate from first principles reflects our current ignorance and remains a challenging bottleneck to the understanding of strongly interacting quark gluon plasma in non Abelian gauge theories.

APPENDIX A: TRACE ANOMALY AT FINITE TEMPERATURE

In this appendix we review the derivation of the trace anomaly at finite temperature [15, 22, 23, 25, 26]. In the Euclidean path integral representation, the partition function for gluodynamics reads

$$Z = \int \mathcal{D}\bar{A}_{\mu,a} \exp\left[-\frac{1}{4g^2}\int d^4x (\bar{G}_{\mu\nu}^a)^2\right] \qquad (A1)$$

where the field strength tensor is given by $\bar{G}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu - i[\bar{A}_\mu, \bar{A}_\nu]$ and $(\bar{G}_{\mu\nu}^a)^2 > 0$ in Euclidean space. Gauge fixing terms will be considered in the gluonic measure $\mathcal{D}\bar{A}_{\mu,a}$, with periodic boundary conditions. The canonical gluon fields are given by $A_\mu = \bar{A}_\mu/g$. This provides the relation

$$\frac{\partial \log Z}{\partial g} = \frac{1}{2g^3}\left\langle \int d^4x (\bar{G}_{\mu\nu}^a)^2 \right\rangle = \frac{1}{2g}\frac{V}{T}\langle (G_{\mu\nu}^a)^2 \rangle \ (A2)$$

where in the last equality translational invariance has been used.

From the standard thermodynamic relations, the free energy, the pressure and energy density are given by

$$F = -PV = -T \log Z \qquad (A3)$$

$$\epsilon = \frac{E}{V} = \frac{T^2}{V}\frac{\partial \log Z}{\partial T} \qquad (A4)$$

as well as the relation

$$\epsilon - 3P = T^5 \frac{\partial}{\partial T}\left(\frac{P}{T^4}\right). \qquad (A5)$$

The dimensionless quantity P/T^4 is a function of T/Λ_{QCD} (or rather $\Lambda_{\text{gluodynamics}}$ in our case) and in principle it can be expressed to all orders in terms of the coupling constant and the scale:

$$\frac{P}{T^4} = f(g(\mu), \log(\mu/2\pi T)). \qquad (A6)$$

The right-hand side is actually independent of the scale μ. This allows to compensate a variation in T with a variation in μ and then in g (i.e., in Λ_{QCD})

$$\frac{\partial}{\partial \log T}\left(\frac{P}{T^4}\right) = \frac{\partial g}{\partial \log \mu}\frac{\partial}{\partial g}\left(\frac{P}{T^4}\right) \qquad (A7)$$

Using now

$$\frac{P}{T^4} = \frac{\log Z}{VT^3} \tag{A8}$$

with (A2) and (A5) yields finally the trace anomaly,

$$\epsilon - 3P = \frac{\beta(g)}{2g} \langle (G_{\mu\nu}^a)^2 \rangle , \tag{A9}$$

where we have introduced the beta function

$$\beta(g) = \mu \frac{dg}{d\mu} . \tag{A10}$$

In this work we focus on the evaluation of the trace anomaly density above the phase transition, which in practice means momenta above $\mu \sim 1\,\mathrm{GeV}$ if the customary $\overline{\mathrm{MS}}$ conversion factor $\mu = 2\pi T$ is adopted. In perturbation theory the beta function reads

$$\beta(g) = -\beta_0 g^3 + \mathcal{O}(g^5) \tag{A11}$$

where $\beta_0 = 11N_c/(48\pi^2)$.

The result for the trace anomaly density to two loops is [58]

$$\frac{\epsilon - 3P}{T^4} = \frac{N_c(N_c^2 - 1)}{72}\beta_0 g^4(T) + \mathcal{O}(g^5) \tag{A12}$$

where $1/g^2(\mu) = \beta_0 \log(\mu^2/\Lambda_{\mathrm{QCD}}^2)$ at leading order. Remarkably this perturbative result implies a negative expectation value of the squared field strength tensor; this quantity being positive prior to renormalization. Note the ambiguity in the (truncated) perturbative result, since generally one has both the temperature T and the $\overline{\mathrm{MS}}$-renormalization scale μ, for which it is usually taken in the literature the reasonable but *arbitrary* choice $\mu \sim 2\pi T$. This issue is analyzed in section IV. Higher order corrections including up to $g^6 \log g$ can be traced from [51]. Unfortunately severe infrared problems in the perturbative expansion yield poor convergence at temperatures available to lattice QCD calculations, $T < 5\,T_c$.

APPENDIX B: THE GLUEBALL HAGEDORN SPECTRUM

The known glueball spectrum as obtained from the lattice ends at a maximum mass value M_{max}. Although this is somewhat tangential to the subject of the present paper it is interesting to analyze the effects due to a possible contribution of higher mass states. The density of states is defined as

$$\rho(M) = \sum_i g_i \delta(M - M_i) = \frac{dN(M)}{dM}, \tag{B1}$$

where $N(M)$ is the cumulative number of glueball states

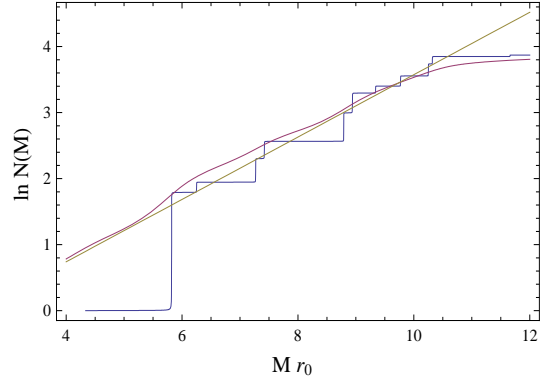$$N(M) = \sum_i g_i \Theta(M - M_i). \tag{B2}$$

FIG. 7: Logarithmic plot of the cumulative number of glueball states obtained from the lattice as a function of the mass in units of the Sommer parameter r_0 compared to the exponential spectrum fit $N(M) = Ae^{M/T_H}$ (straight line) with $T_H = 2.1/r_0$. The smoothed cumulative number with $r_0\Delta M = 0.5$ is also shown.

Here we conventionally take $\Theta(0) = 1/2$. A logarithmic plot of $N(M)$ for the glueball spectrum is presented in Fig. 7 where a rough straight line can be envisaged with the exception of low lying states and the highest states, presumably due to boundary effects.[3] We fit the function

$$N(M) = Ae^{M/T_H} \tag{B3}$$

by minimizing

$$\chi^2 = \sum_i \left(\log N(M_i) - \log(Ae^{M_i/T_H}) \right)^2 . \tag{B4}$$

This gives $r_0 T_H = 2.1$ or equivalently $T_H = 2.8\,T_c$. A weak point of the present determination is the inability to provide a reliable error estimate for the former fit, as there are fluctuations in the spectrum that our smooth fitting function can never account for. To consider this possibility we also represent the *smoothed* cumulative number

$$\langle N_{\mathrm{lat}}(M) \rangle = \sum_i g_i \left(\frac{1}{\pi} \tan^{-1}\left[\frac{M - M_i}{\Delta M} \right] + \frac{1}{2} \right) \tag{B5}$$

which mimics a Breit-Wigner finite width in the density of states. Fluctuations in the spectrum are washed out already with a common smoothing $r_0\Delta M = 0.5$, a factor only 5 times larger than the uncertainty in the lowest glueball mass. As we can see, the smoothing tends to confirm the fit, suggesting the robustness of the analysis.

---

[3] For the skeptical reader we note that recent updates of the particle spectrum provide a similar pattern for both mesons and baryons separately [68, 69].

APPENDIX C: PERTURBATIVE EXPANSION OF THE FREE ENERGY

We compute in this appendix the perturbative expansion of the free energy of a hot gluon plasma using a renormalization group (RG) improvement of the series. The pressure of gluodynamics has been calculated within the weak coupling expansion through $\mathcal{O}(g^6 \log g)$ in Ref. [43, 45, 56, 57, 58, 59, 60, 61], up to an unknown constant. The result can be written in the form

$$
\begin{aligned}
H_{\text{pert}} =\ & 1 + a_2 \alpha + a_3 \alpha^{3/2} \\
& + \alpha^2 \left[a_4 + b_4 \log\left(\frac{\mu}{2\pi T}\right) + c_4 \log \alpha \right] \\
& + \alpha^{5/2} \left[a_5 + b_5 \log\left(\frac{\mu}{2\pi T}\right) \right] \\
& + \alpha^3 \left[a_6 + b_6 \log\left(\frac{\mu}{2\pi T}\right) + c_6 \log \alpha \right. \\
& \left. + d_6 \log\left(\frac{\mu}{2\pi T}\right) \log \alpha + e_6 \log^2\left(\frac{\mu}{2\pi T}\right) \right] \\
& + \mathcal{O}(\alpha^{7/2}),
\end{aligned} \tag{C1}
$$

where we have defined $H_{\text{pert}} = P_{\text{pert}}/P_{\text{ideal}}$, being $P_{\text{ideal}} = ((N_c^2 - 1)\pi^2/45)T^4$ the pressure of an ideal gas of massless gluons. In this formula $\alpha = \alpha(\mu) = g^2(\mu)/4\pi$ is the running coupling constant. All coefficients appearing in (C1) are known except a_6, which cannot be computed within the perturbative scheme owing to infrared divergencies [62].

The trace anomaly follows straightforwardly from the expression of $((N_c^2 - 1)\pi^2/45)H_{\text{pert}}$ by applying $T\partial/\partial T$ (the derivative affects only the explicit T dependence).

We can see in Eq. (C1) that $P_{\text{pert}}(T)$ is computed in the weak coupling limit as an expansion in powers of $\alpha(\mu)$ with coefficients which depend on $\log(\mu/2\pi T)$. However the dependence in μ disappears when all orders are added

$$
\mu \frac{d}{d\mu} H_{\text{pert}} = 0. \tag{C2}
$$

This is because not all coefficients a_i, b_i, c_i, are independent. Indeed

$$
\mu \frac{\partial \alpha(\mu)}{\partial \mu} = \gamma(\alpha) \tag{C3}
$$

with

$$
\gamma(\alpha) = -\alpha^2 \left(\gamma_0 + \gamma_1 \alpha + \gamma_2 \alpha^2 + \cdots \right) \tag{C4}
$$

and $\gamma_0 = \frac{22}{4\pi}$, $\gamma_1 = \frac{204}{(4\pi)^2}$ and $\gamma_2 = \frac{2857}{(4\pi)^3}$ for $N_c = 3$. When this is applied to Eq. (C1) it implies

$$
\begin{aligned}
b_4 &= \gamma_0 a_2, \\
b_5 &= \frac{3}{2} \gamma_0 a_3, \\
b_6 &= \gamma_0 (2a_4 + c_4) + \gamma_1 a_2, \\
d_6 &= 2\gamma_0 c_4, \\
e_6 &= \gamma_0^2 a_2 .
\end{aligned} \tag{C5}
$$

Of course, RG invariance is spoiled if only a finite number of terms is retained. For the truncated series to be useful, $\alpha(\mu)$ must be small, i.e., $\mu \gg \Lambda_{\text{QCD}}$, with fixed $2\pi T/\mu$, which in turn requires $T \gg \Lambda_{\text{QCD}}$.

We can go further and try to reorganize the series of H_{pert} in such a way that it appears explicitly RG invariant. To this end we need to identify some invariant quantity and make the expansion around it. The solution of Eq. (C3) can be written as

$$
-\int_{\alpha(\mu)}^{\alpha(\mu_0)} \frac{d\alpha}{\gamma(\alpha)} = \log\left(\frac{\mu}{\mu_0}\right) . \tag{C6}
$$

Defining, as usual, Λ_{QCD} by the condition $\alpha(\Lambda_{\text{QCD}}) = \infty$, implies

$$
\begin{aligned}
-\int_{\alpha(\mu)}^{\infty} \frac{d\alpha}{\gamma(\alpha)} &= \log\left(\frac{\mu}{\Lambda_{\text{QCD}}}\right) \\
&= \log\left(\frac{2\pi T}{\Lambda_{\text{QCD}}}\right) + \log\left(\frac{\mu}{2\pi T}\right).
\end{aligned} \tag{C7}
$$

This allows to express $\alpha(\mu)$, appearing in the left-hand side, in terms of $\log(\Lambda_{\text{QCD}}/2\pi T)$ and $\log(\mu/2\pi T)$. For convenience, we introduce the quantity

$$
\tilde{\alpha}(T) = \frac{1}{\gamma_0} \frac{1}{\log\left(\frac{2\pi T}{\Lambda_{\text{QCD}}}\right)} , \tag{C8}
$$

which is manifestly RG invariant and behaves as $\alpha(\mu)$ when $\mu = 2\pi T \to \infty$. From (C7) and (C4) then

$$
\begin{aligned}
\alpha(\mu) =\ & \tilde{\alpha} + \tilde{\alpha}^2 \left[\frac{\gamma_1}{\gamma_0} \log \tilde{\alpha} - \gamma_0 \log\left(\frac{\mu}{2\pi T}\right) \right] \\
& + \tilde{\alpha}^3 \left[\frac{\gamma_2}{\gamma_0} - \frac{\gamma_1^2}{\gamma_0^2} + \frac{\gamma_1^2}{\gamma_0^2} \log \tilde{\alpha} + \frac{\gamma_1^2}{\gamma_0^2} \log^2 \tilde{\alpha} \right. \\
& \left. - \gamma_1 \log\left(\frac{\mu}{2\pi T}\right) + \gamma_0^2 \log^2\left(\frac{\mu}{2\pi T}\right) \right. \\
& \left. - 2\gamma_1 \log \tilde{\alpha} \log\left(\frac{\mu}{2\pi T}\right) \right] \\
& + \mathcal{O}(\tilde{\alpha}^4) .
\end{aligned} \tag{C9}
$$

By introducing this perturbative series of $\alpha(\mu)$, in Eq. (C1) one obtains

$$
\begin{aligned}
H_{\text{pert}} =\ & 1 + A_2 \tilde{\alpha} + A_3 \tilde{\alpha}^{3/2} \\
& + \tilde{\alpha}^2 \left[A_4 + B_4 \log \tilde{\alpha} \right] \\
& + \tilde{\alpha}^{5/2} \left[A_5 + B_5 \log \tilde{\alpha} \right] \\
& + \tilde{\alpha}^3 \left[A_6 + B_6 \log \tilde{\alpha} + C_6 \log^2 \tilde{\alpha} \right] \\
& + \mathcal{O}(\tilde{\alpha}^{7/2}).
\end{aligned} \tag{C10}
$$

All coefficients here are known except A_6. In fact[4]

$$
A_2 = a_2,
$$

---

[4] The values for the coefficients are $A_2 = -15/(4\pi)$, $A_3 = 30/\pi^{3/2}$, $A_4 = 135(3.51 - \log \pi)/(2\pi^2)$, $B_4 = 5175/(88\pi^2)$, $A_5 = -3.23 * 495/(2\pi^{5/2})$, $B_5 = 2295/(22\pi^{5/2})$, $B_6 = 2295/(1936\pi^3)(115 + 264(3.51 - \log \pi)) - 1134.8/\pi^3$, $C_6 = 566865/(1936\pi^3)$.

$$A_3 = a_3,$$
$$A_4 = a_4,$$
$$B_4 = \frac{\gamma_1}{\gamma_0}a_2 + c_4,$$
$$A_5 = a_5,$$
$$B_5 = \frac{3}{2}\frac{\gamma_1}{\gamma_0}a_3,$$
$$A_6 = \left(\frac{\gamma_2}{\gamma_0} - \frac{\gamma_1^2}{\gamma_0^2}\right)a_2 + a_6,$$
$$B_6 = \frac{\gamma_1^2}{\gamma_0^2}a_2 + 2\frac{\gamma_1}{\gamma_0}a_4 + \frac{\gamma_1}{\gamma_0}c_4 + c_6,$$
$$C_6 = \frac{\gamma_1^2}{\gamma_0^2}a_2 + 2\frac{\gamma_1}{\gamma_0}c_4. \qquad \text{(C11)}$$

H_{pert} is now written in a manifestly RG invariant way as a function of T/Λ_{QCD}. In principle, the parameter Λ_{QCD} itself can be extracted from the pressure at suffi-ciently high temperature:

$$\log \Lambda_{\text{QCD}} = \frac{1}{\gamma_0 A_2}\lim_{T\to\infty}\left(\frac{H_{\text{pert}} - 1}{t^2} - \frac{A_2}{t}\right.$$
$$\left. -\frac{A_3}{t^{1/2}} - B_4 \log t - A_4\right), \qquad \text{(C12)}$$

where $t = 1/(\gamma_0 \log(2\pi T))$ (and Λ_{QCD} results in the same units used for T in t). In practice, the pressure is not known to the required precision to such high temperatures and the parameter Λ_{QCD} is obtained from other determinations.

The trace anomaly follows immediately from the RG expression noting that $d\tilde\alpha/d\log T = -\gamma_0\tilde\alpha^2$,

$$\Delta_{\text{pert}} = -\frac{(N_c^2 - 1)\pi^2}{45}\gamma_0\tilde\alpha^2\frac{dH_{\text{pert}}}{d\tilde\alpha}, \qquad \text{(C13)}$$

known modulo $\mathcal{O}(\tilde\alpha^{9/2})$.

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